Compositional Equivalence Checking of Imperative Programs: A Game-Semantic Approach

Luke Ong

Oxford University Computing Laboratory

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Model checking: Extremely successful in verifying finite-state processes. E.g. digital circuits and communication protocols.

Over the past decade, huge strides made in **verification of 1st-order imperative programs**. Many tools: SLAM, Blast, SatAbs, etc.

State-of-the-art tools use abstraction techniques, as exemplified by CEGAR (Counter-Example Guided Abstraction Refinement), and acceleration methods such as SAT- and SMT-solvers.

An Alternative Approach

Start from an accurate denotational semantics of the program; then derive an appropriate model of computation sufficiently concrete (and tractable) for verification.

Advantages: Soundness and completeness inherited by the model; method remains compositional.

Is there such a semantics?

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[Abramsky, Jagadeesan, Malacaria, Hyland, O., Hanno, McCusker, etc.]

Game semantics has emerged as a powerful paradigm for giving semantics to a wide range of programming languages (procedural, higher-order functional, polymorphic, reference types, non-local control, concurrent, probabilistic, etc.).

These models are highly accurate (fully abstract).

Promising features of game semantics

- Clear operational content, while admitting compositional methods in the style of denotational semantics.
- Strategies are highly-constrained processes, admitting automata-theoretic representations.
- Rich mathematical structures yielding accurate models of advanced high-level programming languages.

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Challanges of the Approach

To carry over methods of model checking to much more structured, modern programming situations, in which the following features are important:

- data-types: references (pointers), recursive types
- non-local control flow: exceptions, call-cc, etc.
- modularity principles: e.g. object orientation: inheritance and subtyping
- higher-order features: higher-order procedures; closures; components
- variables and names: passing mechanisms, life-span, scoping rules
- concurrency and non-determinism: synchronization, multithreading, etc.

Aim:

Combine results and insights in (game) semantics, with techniques in verification.

Outline

1. Idealized Algol and Observational Equivalence
2. Game Semantics: An Impressionistic Introduction
3. Using Game Semantics to Decide Observational Equivalence
4. Homer: **H**igher-order **O**bservational-equivalence **M**odel check**ER**

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A compact language that elegantly combines state-based procedural and higher-order functional programming, using a simple type-theoretic framework. IA is essentially a call-by-name variant of Core ML.

IA Types:

$$T ::= \mathsf{exp} \mid \mathsf{com} \mid \mathsf{var} \mid T \to T$$

$T ::=$	exp	numbers-valued expressions
$\mid$	com	commands
$\mid$	var	assignable variables
$\mid$	$T \to T$	function space

IA Terms:

- imperative constructs
- block-allocated local assignable variables
- PCF (= simply-typed λ-calculus + basic arithmetics + conditionals + fixpoint operators).

In this talk, we suppress higher-order features, though not completely. (E.g. Recursive 1st-order procedures are fixpoints of 2nd-order functionals.)

$\mathrm{ord}(o) := 0 \qquad \mathrm{ord}(A \to B) := \max(\mathrm{ord}(A) + 1, \mathrm{ord}(B))$

Idealized Algol (IA) [Reynolds 80]

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Examples

```
x : exp |- new X in
               new Y in
                   X := x;
                   Y := 1;
                   while !X > 0 do
                   {
                         Y := !Y * !X;
                         X := !X - 1
                   };
                   !Y
```

Notation. Assignable variables ranged over by X, Y, etc.

```
|- fun f : (exp -> exp) -> exp .
             f (fun x : exp . f (fun y : exp . x))
```

$$\lambda f.f\,(\lambda x.f\,(\lambda y.x)) : ((\mathrm{exp} \to \mathrm{exp}) \to \mathrm{exp}) \to \mathrm{exp}$$

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Observational (or Contextual) Equivalence

[Milner 1975, Plotkin 1977, ... Full Abstraction Problem for PCF]
Intuitively $M \approx N$ means

> *"M and N are mutually substitutable in every program context without causing any difference in the computational outcome".*

Definition $M \approx N$ just if for every context $C[\,]$ such that $C[M]$ and $C[N]$ are programs (i.e. closed terms of base type), for every value v

$$C[M] \Downarrow v \quad \Longleftrightarrow \quad C[N] \Downarrow v.$$

- Quantification over all program contexts $C[-]$ ensures that potential side effects of M and N are taken fully into account.
- $\approx$ is an intuitively compelling notion of program equivalence, but very hard to reason about.
- An appropriate notion of equivalence for regression verification, for maintaining backwards compatibility of code. (Cf. Strichman's lecture)

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Example 1: In Algol-like languages, state changes are irreversible.

I.e. "Snap-back", a construct

$$\text{Snapback} \quad : \quad \text{com} \rightarrow \text{com}$$

that runs its command-argument and then immediately undoes all the state-changes caused by the command, is not definable in IA.

Non-definability of snap-back is equivalent to:

$$\begin{array}{ll} & p : \text{com} \longrightarrow \text{com} \\ \vdash & \text{new}\, X := 0 \,\text{in}\, \{p\ (X := 1)\, \mathbf{;}\, \text{if}\ \mathbf{!}X = 1\ \text{then}\ \mathbf{\Omega}\ \text{else}\,\text{skip}\} \\ \approx & p\ \mathbf{\Omega} \end{array}$$

The theory of observational equivalence is rich

Example 2: Parametricity

Terms that have the "same underlying algorithm" are observationally equivalent.

$$
\begin{array}{ll}
 & p : \mathsf{com} \to \mathsf{bool} \to \mathsf{com} \\
\vdash & \mathsf{new}\, X := 1 \,\mathsf{in}\, \{p\,(X := -!X)\,(!X > 0)\} \\
\approx & \mathsf{new}\, Y := \mathsf{t} \,\mathsf{in}\, \{p\,(Y := \neg !Y)\,(!Y)\}
\end{array}
$$

IA is Turing powerful: observational equivalence is not decidable.

Questions

1. For which fragment of IA is observational equivalence decidable?
2. Classify these fragments.

Game semantics helps to answer these questions.

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Game Interpretation of Types and Programs: Some Generalities

Types of a programming language are interpreted as (2-person) games.

Player	Point of View	
P (Proponent)	System	Term being modelled
O (Opponent)	Environment	Program context

Programs are interpreted as strategies for playing these games.

Game semantics is inherently a semantics of open systems; the meaning of a program is given by its potential interactions with the environment.

Compositionality: The key operation is plugging two strategies together, so that each actualizes part of the environment of the other.

$$\frac{\sigma : A \longrightarrow B \qquad \tau : B \longrightarrow C}{\sigma ; \tau : A \longrightarrow C}$$

This exploits the P/O duality: σ's P-move at B become an O-move of τ (and vice versa).

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Innocent Game or HON Game: Some Intuitions

[Hyland+O. (1995) Info & Comp 2000; Nickau 96. Precursors: Lorenz and Lorenzen, Conway, Joyal, Blass, Berry + Curien.]

Play as dialogue between O and P

Four types of move: P-questions, O-answers, O-questions, P-answers.

A play is an O/P-alternating sequence of moves, satisfying:

Rules of Civil Conversation

Justification:

- A question is asked only if the dialogue warrants it at that point.
- An answer is proferred only if a question expecting it is pending.

Well-Bracketing: "Last asked first answered."

Outcome of play: A dialogue ends when the opening question is answered. (We don't care about winning: just play to the bitter end!)

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Take $\vdash M : A$.

- Type A is interpreted as (a 2-player game called) arena $[\![A]\!]$.
- Term M is interpreted as a P-strategy $[\![M]\!]$ for playing in arena $[\![A]\!]$.

An arena is a forest (the nodes are the moves; edge relation is called enabling); each move has a label from $\{ PQ, PA, OQ, OA \}$.

Example. The arena $[\![\text{exp}]\!]$

OQ

PA

$[\![2 : \text{exp}]\!]$ is the P-strategy:

OQ	q
PA	2

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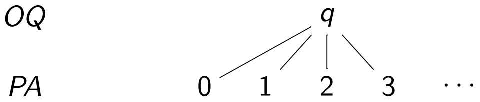

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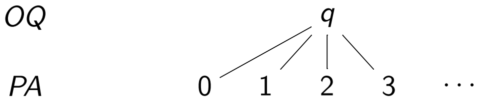

$[\![2 : \mathrm{exp}]\!]$ is the P-strategy:

OQ	q
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Write "if B then M else N" in prefix form:

$$\text{if } B\ M\ N : \text{exp}$$

- if : exp → exp → exp → exp is interpreted as a P-strategy
- program context $[\,]\ B\ M\ N$ determines an O-strategy

for playing in the arena $[\![\text{exp} \to \text{exp} \to \text{exp} \to \text{exp}]\!]$

	B		M		N		
	exp	→	exp	→	exp	→	exp
OQ							q
PQ	q						
OA	t						
PQ			q				
OA			3				
PA							3

(assuming O-strategy given by $[\,]\ \text{t}\ 3\ 4$).

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for playing in the arena $[\![\, \text{exp} \rightarrow \text{exp} \rightarrow \text{exp} \rightarrow \text{exp} \,]\!]$

	B exp	$\rightarrow$	M exp	$\rightarrow$	N exp	$\rightarrow$	exp
OQ							q
PQ	q						
OA	f						
PQ					q		
OA					4		
PA							4

(assuming O-strategy given by $[\,]\ \text{f}\ 3\ 4$).

The arena $[\![\mathsf{com}]\!]$

OQ	run
PA	$done$

$[\![\mathsf{skip} : \mathsf{com}]\!]$ is the P-strategy:

OQ	run
PA	$done$

Interpreting sequential composition: $\mathbf{;} : \mathsf{com} \to \mathsf{com} \to \mathsf{com}$

	com	→	com	→	com
OQ					run
PQ	run				
OA	$done$				
PQ			run		
OA			$done$		
PA					$done$

The arena $[\![\, \mathrm{com} \,]\!]$

OQ	*run*
PA	*done*

$[\![\, \mathrm{skip} : \mathrm{com} \,]\!]$ is the P-strategy:

OQ	*run*
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Interpreting sequential composition: **;** $: \mathrm{com} \rightarrow \mathrm{com} \rightarrow \mathrm{com}$

	com	$\rightarrow$	com	$\rightarrow$	com
OQ					*run*
PQ	*run*				
OA	*done*				
PQ			*run*		
OA			*done*		
PA					*done*

The arena $[\![\mathrm{com}]\!]$

$$\begin{array}{cc} OQ & run \\ & | \\ PA & done \end{array}$$

$[\![\mathrm{skip} : \mathrm{com}]\!]$ is the P-strategy:

$$\begin{array}{cc} OQ & run \\ PA & done \end{array}$$

Interpreting sequential composition: $\mathbf{;} : \mathrm{com} \to \mathrm{com} \to \mathrm{com}$

$$\begin{array}{ccccccc} & \mathrm{com} & \to & \mathrm{com} & \to & \mathrm{com} \\ OQ & & & & & run \\ PQ & run & & & & \\ OA & done & & & & \\ PQ & & & run & & \\ OA & & & done & & \\ PA & & & & & done \end{array}$$

Interpreting var

Following Reynolds, we view a variable type as given (in an object-oriented style) by a product of its read method and its write method. Thus

$$\mathsf{var} \quad := \quad \mathsf{exp} \times (\prod_{i \in \omega} \mathsf{com})$$

- read-part: first component is the value held at that location
- write-part: second component contains countably many commands, namely, to write 0 (respectively 1, 2, etc.) to that location.

Thus arena $[\![\, \mathsf{var} \,]\!]$ is the product arena $[\![\, \mathsf{exp} \,]\!] \times \prod_{i \in \omega} [\![\, \mathsf{com} \,]\!]$:

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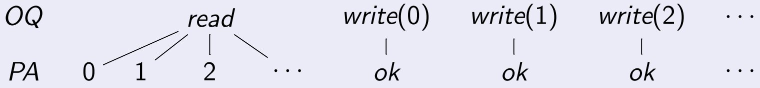

Write $X := M$ as **assign** $X\ M$. Thus

- **assign** : var → exp → com is interpreted as a P-strategy
- context $[\,]\ X\ M$ determines an O-strategy

for playing in the arena $[\![$ var → exp → com $]\!]$.

	X		M		
	var	→	exp	→	com
OQ					run
PQ			q		
OA			5		
PQ	$write(5)$				
OA	ok				
PA					$done$

(assuming O-strategy is given by context $[\,]\ X\ 5$)

Interpreting assignment: $X := M$

Write $X := M$ as **assign** $X\ M$. Thus

- **assign** : var $\rightarrow$ exp $\rightarrow$ com is interpreted as a P-strategy
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	X var	$\rightarrow$	M exp	$\rightarrow$	com
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PQ	$write(5)$				
OA	ok				
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(assuming O-strategy is given by context $[\,]\,X\ 5$)

Interpreting block-allocated local variables: new

We decompose the formation

$$\frac{\Gamma, x : \mathsf{var} \vdash M : \mathsf{com}}{\Gamma \vdash \mathsf{new}\, x := n\, \mathsf{in}\, M : \mathsf{com}}$$

into two constructions:

1. Currying: $\Gamma \vdash \boldsymbol{\lambda} x : \mathsf{var}.M : \mathsf{var} \to \mathsf{com}$
2. Application by a constant: $\mathbf{new}_n : (\mathsf{var} \to \mathsf{com}) \to \mathsf{com}$.

Thus we have

$$\mathsf{new}\, x := n\, \mathsf{in}\, M \;:=\; \mathbf{new}_n\, (\boldsymbol{\lambda} x : \mathsf{var}.M)$$

Accordingly $[\![\, \mathsf{new}\, x := n\, \mathsf{in}\, M \,]\!]$ is the composite

$$[\![\, \Gamma \,]\!] \xrightarrow{[\![\, \Gamma \vdash \boldsymbol{\lambda} x:\mathsf{var}.M \,]\!]} (\mathsf{var} \to \mathsf{com}) \xrightarrow{\mathbf{new}_n} \mathsf{com}$$

Question. What is the strategy $\mathbf{new}_n$?

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Question. What is the strategy $\mathbf{new}_n$?

The plays in $\mathbf{new}_n$ should correspond to the behaviour of a *prima facie* variable (initialized to n). Namely, they should satisfy:

Good Variable Property

Whenever the variable is read, it yields the value last written to it.

Thus, the (maximal) plays are defined to be words matching the regular expression:

$$q \cdot q^{\langle 1 \rangle} \cdot (read \cdot n)^* \cdot \left(\sum_{i \geq 0} write(i) \cdot ok \cdot (read \cdot i)^* \right)^* \cdot done^{\langle 1 \rangle} \cdot done$$

The (infinite) alphabet is the move-set of $(\mathsf{var} \to \mathsf{com}^{\langle 1 \rangle}) \to \mathsf{com}$ (subject to the labelling convention to distinguish copies of the same subarena).

Good Variable Behaviour: An Example Play in $\mathbf{new}_n$

	(var	$\rightarrow$	$\mathsf{com}^{\langle 1\rangle}$)	$\rightarrow$	com
OQ					run
PQ			$run^{\langle 1\rangle}$		
OQ	$read$				
PA	n				
OQ	$write(5)$				
PA	ok				
OQ	$read$				
PA	9				
OA			$done^{\langle 1\rangle}$		
PA					$done$

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Finitary Idealized Algol IA_f: recursion-free, finite base types

Recall: $\mathrm{ord}(b) := 0 \quad \mathrm{ord}(T_1 \to T_2) := \max(\mathrm{ord}(T_1)+1, \mathrm{ord}(T_2))$

An IA_f-term $\quad x_1 : T_1, \cdots, x_n : T_n \vdash M : T \quad$ is an ***i*-th order term** just if $\mathrm{ord}(T_j) < i$ and $\mathrm{ord}(T) \le i$.

- IA_i: collection of *i*-th order IA_f-terms.
- IA_i+**while** is IA_i augmented by while-loops
- $\mathsf{IA}_i+\mathbf{Y}_j$ **(where $j < i$)** is IA_i augmented by

$$\frac{\Gamma, f : T \vdash M : T}{\Gamma \vdash \mu f^T.M : T}$$

where the premise is *i*-th order, and $\mathrm{ord}(T) \le j$.
I.e. $\mathsf{IA}_i+\mathbf{Y}_j$ consists of IA_i and recursively-defined terms of order at most j.

Theorem (Full Abstraction, Abramsky + McCusker 1997)

Observational equivalence of IA is characterized by complete plays (i.e. plays ending with a move that answers the opening question):

$$M \approx N \iff \mathbf{cplays}([\![\Gamma \vdash M : A]\!]) = \mathbf{cplays}([\![\Gamma \vdash N : A]\!])$$

At low types, game semantics admits a concrete representation.

Theorem (Ghica + McCusker 2000)

In IA_2+**while**:

1. $\mathbf{cplays}([\![\Gamma \vdash M : A]\!])$ *is regular. Further*
2. $\mathbf{cplays}([\![\Gamma \vdash M : A]\!])$*, given as a DFA (or regular expression), can be constructed by recursion over syntax.*

Hence $\approx$ *in* IA_2 *reduces to the problem of DFA-equivalence.*

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1. $\mathbf{cplays}(\llbracket \Gamma \vdash M : A \rrbracket)$ *is regular. Further*
2. $\mathbf{cplays}(\llbracket \Gamma \vdash M : A \rrbracket)$*, given as a DFA (or regular expression), can be constructed by recursion over syntax.*

Hence $\approx$ *in* IA_2 *reduces to the problem of DFA-equivalence.*

A Complete Classification of Decidable Fragments of Finitary IA

OBS EQUIV$_L$: Given β-nfs M and N in sublanguage L of IA, does $M \approx N$?

	pure	+**while**	$+\mathbf{Y}_0$	$+\mathbf{Y}_1$
IA_0	PTIME	–	–	–
IA_1	coNP	PSPACE	DPDA EQUIV	undecidable
IA_2	PSPACE	PSPACE	DPDA EQUIV	undecidable
IA_3	EXPTIME	EXPTIME	DPDA EQUIV	undecidable
$\mathsf{IA}_i, i \geq 4$	undecidable	undecidable	undecidable	undecidable

Undecidability results: O. LICS'02 and Murawski LICS'03.
coNP + PSPACE results: Murawski TCS 2005.
EXPTIME results: O. LICS'02; Murawski + Walukiewicz FOSSACS'05.
DPDA EQUIV results: Murawski, Walukiewicz + O. ICALP'05.

Visibly pushdown automata (Alur+Madhusudan, STOC'04)

- Stack action is determined by input alphabet read:
  $\Sigma = \Sigma_{\mathrm{push}} + \Sigma_{\mathrm{pop}} + \Sigma_{\mathrm{noop}}$
- Excellent closure properties (almost as good as regular languages)

VPA-languages: Closed under complementation and intersection (*cf.* DPDA).

$$L(A) \subseteq L(B) \iff L(A) \cap \overline{L(B)} = \varnothing$$

is EXPTIME-complete, and in PTIME if B deterministic.

Theorem (Murawski+Walukiewicz 2005)

*The complete plays of (IA_3+**while**)-terms are VPA-recognizable.*

EXPTIME-hardness: by reducing the EXPTIME-complete problem FINITE TREE AUTOMATA EQUIVALENCE (Seidl 1990) to it.

[Murawski, O. + Walukiewicz ICALP'05]

$\mathsf{IA}_i+\mathbf{Y}_0$: Only terms of base type can call themselves recursively. This includes all tail-recursive functions (i.e. iterations) and:

Example. Non tail-recursive ground recursion:

$$c : \mathsf{com}, b : \mathsf{bool} \vdash \mu p^{\mathsf{com}}.\mathsf{if}\ b\ \mathsf{then}\ (p\ ;\ c\ ;\ p)\ \mathsf{else}\ \mathsf{skip} : \mathsf{com}$$

DPDA-Equivalenc hardness

Theorem

There is a translation that maps a DPDA A to a $(\mathsf{IA}_1+\mathbf{Y}_0)$-term $x : \mathsf{exp} \vdash M_A : \mathsf{com}$ such that for any A, B, we have $L(A) = L(B)$ iff $M_A \approx M_B$.

Outline

1. Idealized Algol and Observational Equivalence
2. Game Semantics: An Impressionistic Introduction
3. Using Game Semantics to Decide Observational Equivalence
4. Homer: **H**igher-order **O**bservational-equivalence **M**odel check**ER**

HOMER: Higher-order Observational-equivalence Model checkER

[Hopkins + O. CAV 2009]

HOMER: a prototype tool implementing Murawski and Walukiewicz's algorithm.

HOMER maps IA_3+**while** terms to VPA representing the complete plays in their game semantics, then check for the equivalence of the VPA.
If the input term is at most 2nd-order (possibly with iteration), the VPA-compile is just a DFA.

Counterexample. If the terms are inequivalent, HOMER will produce both a game-semantic and an operational-semantic counterexample, in the form of a play and a separating context respectively.

Property checking. HOMER can also model check a term against a regular property or LTL formula.

HOMER is written in about 8 KLOC of F#, including about 600 LOC for the VPA toolkit. It is the first model checker for third-order programs.

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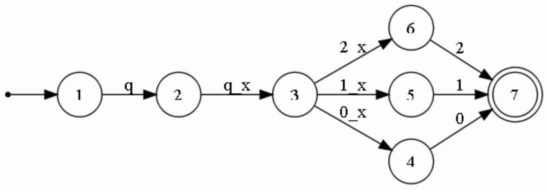
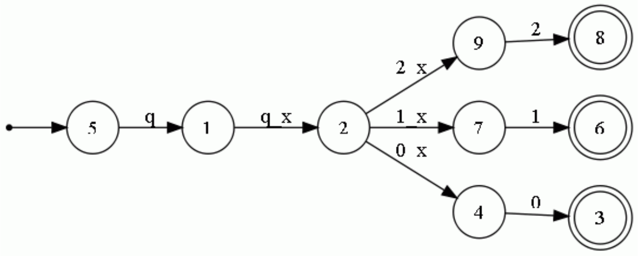
Example 1

```
x :  exp |- x
```

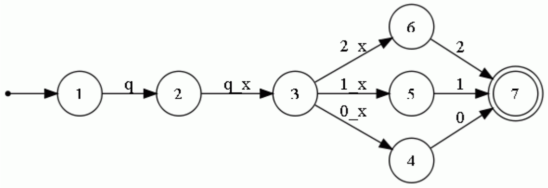



```
x :  exp |- new X in (X := x ; if !X = 0 then !X else !X)
```

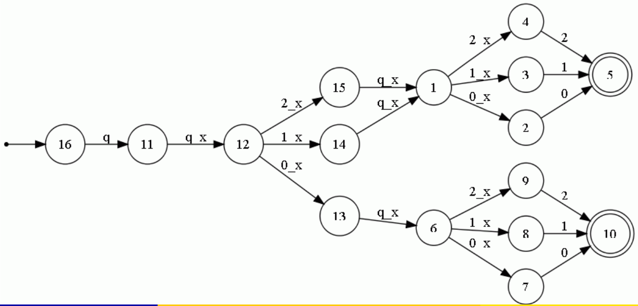


Example 1'

`x :  exp |- x`



`x :  exp |- if x = 0 then x else x`



Why sorting?

"... it seems impossible to use Model Checking to verify that a sorting algorithm is correct since sorting correctness is a data-oriented property involving several quantifications and data structures." [Bandera user manual]

Example: Equivalence of (respective implementations of) bubble sort and insertion sort.

- Program parameterized over array size (n) and basic data type ($\mathbb{Z}$ MOD 3).
- The DFA model is fully abstract. Only the actions of the non-local array are observable, and hence, represented.
- An array of size 20 (over integers MOD 3) has *circa* 3^{20} states (about 3.5 billion). Our model is highly abstract (though still accurate): it has only about 5500 states!

Conclusions and Further Directions

- Game semantics has clear operational content, while admitting compositional methods in the style of denotational semantics.
- The game-semantic approach to observational equivalence checking is fully automatic, sound and complete, and compositional.
- The model (given by complete plays) extracted is highly accurate, yet "compact".
- To extend to infinite data types, use abstraction refinement techniques (Bakewell + Ghica, TACAS08) or prove auxiliary data-independence results.

Further directions

1. Performance: Exploit abstraction refinement techniques (CEGAR), and acceleration technologies (SMT-solvers) to improve scalability.
2. Challenge of verifying highly structured programs (e.g. object orientation / functional features e.g. Javascript, Perl, etc).

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