Compositional Equivalence Checking of Imperative Programs: A Game-Semantic Approach

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Intel Symposium, Technion, 8 Sep 2009
Model checking: Extremely successful in verifying finite-state processes. E.g. digital circuits and communication protocols.

Over the past decade, huge strides made in verification of 1st-order imperative programs. Many tools: SLAM, Blast, SatAbs, etc.

State-of-the-art tools use abstraction techniques, as exemplified by CEGAR (Counter-Example Guided Abstraction Refinement), and acceleration methods such as SAT- and SMT-solvers.

An Alternative Approach

Start from an accurate denotational semantics of the program; then derive an appropriate model of computation sufficiently concrete (and tractable) for verification.

Advantages: Soundness and completeness inherited by the model; method remains compositional.

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Is there such a semantics?
Game semantics has emerged as a powerful paradigm for giving semantics to a wide range of programming languages (procedural, higher-order functional, polymorphic, reference types, non-local control, concurrent, probabilistic, etc.). These models are highly accurate (fully abstract).

Promising features of game semantics

- Clear operational content, while admitting compositional methods in the style of denotational semantics.
- Strategies are highly-constrained processes, admitting automata-theoretic representations.
- Rich mathematical structures yielding accurate models of advanced high-level programming languages.
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- Rich mathematical structures yielding accurate models of advanced high-level programming languages.
Challenges of the Approach

To carry over methods of model checking to much more structured, modern programming situations, in which the following features are important:

- **data-types**: references (pointers), recursive types
- **non-local control flow**: exceptions, call-cc, etc.
- **modularity principles**: e.g. object orientation: inheritance and subtyping
- **higher-order features**: higher-order procedures; closures; components
- **variables and names**: passing mechanisms, life-span, scoping rules
- **concurrency and non-determinism**: synchronization, multithreading, etc.

**Aim:**

Combine results and insights in (game) semantics, with techniques in verification.
Outline

1. Idealized Algol and Observational Equivalence

2. Game Semantics: An Impressionistic Introduction

3. Using Game Semantics to Decide Observational Equivalence

4. Homer: Higher-order Observational-equivalence Model checking
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Idealized Algol (IA) [Reynolds 80]

A compact language that elegantly combines state-based procedural and higher-order functional programming, using a simple type-theoretic framework. IA is essentially a call-by-name variant of Core ML.

**IA Types:**

\[
T ::= \begin{cases} 
\text{exp} & \text{numbers-valued expressions} \\
\text{com} & \text{commands} \\
\text{var} & \text{assignable variables} \\
T \rightarrow T & \text{function space}
\end{cases}
\]

**IA Terms:**

- imperative constructs
- block-allocated local assignable variables
- PCF (= simply-typed $\lambda$-calculus + basic arithmetics + conditionals + fixpoint operators).

In this talk, we suppress higher-order features, though not completely. (E.g. Recursive 1st-order procedures are fixpoints of 2nd-order functionals.)

\[
\text{ord}(\text{o}) := 0 \\
\text{ord}(A \rightarrow B) := \max(\text{ord}(A) + 1, \text{ord}(B)).
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\text{ord}(o) := 0 \quad \text{ord}(A \to B) := \max(\text{ord}(A) + 1, \text{ord}(B))
\]
Examples

\[ x : \text{exp} |- \text{new} \ X \text{ in} \]
\[ \quad \text{new} \ Y \text{ in} \]
\[ \quad X := x; \]
\[ \quad Y := 1; \]
\[ \quad \text{while} \ !X > 0 \text{ do} \]
\[ \quad \{ \]
\[ \quad \quad Y := !Y * !X; \]
\[ \quad \quad X := !X - 1 \]
\[ \quad \}; \]
\[ !Y \]

Notation. Assignable variables ranged over by \( X, Y, \) etc.

\[ |- \text{fun} \ f : (\text{exp} \rightarrow \text{exp}) \rightarrow \text{exp} . \]
\[ f (\text{fun} \ x : \text{exp} . \ f (\text{fun} \ y : \text{exp} . \ x)) \]
\[ \lambda f. f (\lambda x. f (\lambda y. x)) : ((\text{exp} \rightarrow \text{exp}) \rightarrow \text{exp}) \rightarrow \text{exp} \]
Examples

\[\begin{align*}
x : \text{exp} & \vdash \text{new } X \text{ in} \\
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Observational (or Contextual) Equivalence

[Milner 1975, Plotkin 1977, ... Full Abstraction Problem for PCF]

Intuitively $M \approx N$ means

“$M$ and $N$ are mutually substitutable in every program context without causing any difference in the computational outcome”.

**Definition** $M \approx N$ just if for every context $C[\ ]$ such that $C[M]$ and $C[N]$ are programs (i.e. closed terms of base type), for every value $v$

$$C[M] \Downarrow v \iff C[N] \Downarrow v.$$ 

- Quantification over all program contexts $C[-]$ ensures that potential side effects of $M$ and $N$ are taken fully into account.
- $\approx$ is an intuitively compelling notion of program equivalence, but very hard to reason about.
- An appropriate notion of equivalence for regression verification, for maintaining backwards compatibility of code. (Cf. Strichman’s lecture)
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Example 1: In Algol-like languages, state changes are irreversible. 
I.e. “Snap-back”, a construct

\[ \text{Snapback} : \text{com} \rightarrow \text{com} \]

that runs its command-argument and then immediately undoes all the state-changes caused by the command, is not definable in IA.

Non-definability of snap-back is equivalent to:

\[ p : \text{com} \rightarrow \text{com} \]

\[ \vdash \text{new } X := 0 \text{ in } \{ p (X := 1); \text{if } !X = 1 \text{ then } \Omega \text{ else skip} \} \]

\[ \approx \quad p \Omega \]
The theory of observational equivalence is rich

Example 2: Parametricity

Terms that have the “same underlying algorithm” are observationally equivalent.

\[ p : \text{com} \rightarrow \text{bool} \rightarrow \text{com} \]

\[ \vdash \text{new } X := 1 \text{ in } \{ p (X := \neg !X)(!X > 0) \} \]

\[ \approx \text{new } Y := t \text{ in } \{ p (Y := \neg !Y)(!Y) \} \]

IA is Turing powerful: observational equivalence is not decidable.

Questions

1. For which fragment of IA is observational equivalence decidable?
2. Classify these fragments.

Game semantics helps to answer these questions.
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Types of a programming language are interpreted as (2-person) games.

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Programs are interpreted as strategies for playing these games.

Game semantics is inherently a semantics of open systems; the meaning of a program is given by its potential interactions with the environment.

Compositionality: The key operation is plugging two strategies together, so that each actualizes part of the environment of the other.

\[
\sigma : A \rightarrow B \quad \tau : B \rightarrow C \\
\sigma;\tau : A \rightarrow C
\]

This exploits the P/O duality: \( \sigma \)'s P-move at \( B \) become an O-move of \( \tau \) (and vice versa).
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Innocent Game or HON Game: Some Intuitions


Play as dialogue between O and P

Four types of move: P-questions, O-answers, O-questions, P-answers.

A play is an O/P-alternating sequence of moves, satisfying:

Rules of Civil Conversation

Justification:

- A question is asked only if the dialogue warrants it at that point.
- An answer is proferred only if a question expecting it is pending.

Well-Bracketing: “Last asked first answered.”

Outcome of play: A dialogue ends when the opening question is answered. (We don’t care about winning: just play to the bitter end!)
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Basics of Game Semantics by Examples

Take $\vdash M : A$.

- Type $A$ is interpreted as (a 2-player game called) arena $\llbracket A \rrbracket$.
- Term $M$ is interpreted as a P-strategy $\llbracket M \rrbracket$ for playing in arena $\llbracket A \rrbracket$.

An arena is a forest (the nodes are the moves; edge relation is called enabling); each move has a label from $\{ PQ, PA, OQ, OA \}$.

Example. The arena $\llbracket \text{exp} \rrbracket$

$$
\begin{align*}
&\text{OQ} \\
&\text{PA}
\end{align*}
$$

$\llbracket 2 : \text{exp} \rrbracket$ is the P-strategy:

$$
\begin{align*}
&\text{OQ} \\
&\text{q} \\
&\text{PA} \quad 2
\end{align*}
$$
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Example. The arena \( \llbracket \text{exp} \rrbracket \)

\[
\begin{array}{c}
OQ \\
PA \\
\end{array}
\quad
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
\ldots
\end{array}
\]

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2
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\end{array}
\begin{array}{c}
q \\
0 & 1 & 2 & 3 & \ldots
\end{array}
\]

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\text{PA} & 2
\end{array}
\]
Interpreting if-then-else

Write “if $B$ then $M$ else $N$” in prefix form:

$$\text{if } B \ M \ N : \exp$$

- if : $\exp \rightarrow \exp \rightarrow \exp \rightarrow \exp$ is interpreted as a P-strategy
- program context \([ ] B M N\) determines an O-strategy for playing in the arena \([ \exp \rightarrow \exp \rightarrow \exp \rightarrow \exp \] \)

\[
\begin{array}{cccc}
B & M & N \\
\exp & \exp & \exp & \exp \\
OQ & PQ & OA & PQ \\
q & q & t & q \\
PA & 3 & 3 & \\
\end{array}
\]

(assuming O-strategy given by \([ ] t 3 4\).
Write “if $B$ then $M$ else $N$” in prefix form:

```plaintext
if $B M N : \exp$
```

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$\
\begin{align*}
OQ & \rightarrow q \\
PQ & \rightarrow q \\
OA & \rightarrow t \\
PQ & \rightarrow q \\
OA & \rightarrow 3 \\
PA & \rightarrow 3
\end{align*}$

(assuming O-strategy given by $[ \ ] t 3 4$).
Write “if $B$ then $M$ else $N : \text{exp}$” in prefix-form:

\[
\text{if } B \ M \ N : \text{exp}
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- if : $\text{exp} \rightarrow \text{exp} \rightarrow \text{exp} \rightarrow \text{exp}$ is interpreted as a P-strategy.
- Program context $[ ] B M N$ determines an O-strategy for playing in the arena $[ \text{exp} \rightarrow \text{exp} \rightarrow \text{exp} \rightarrow \text{exp} ]$.

\[
\begin{array}{cccc}
B & M & N \\
\text{exp} & \rightarrow & \text{exp} & \rightarrow \\
\text{q}
\end{array}
\]

- $OQ q$
- $PQ q$
- $OA f$
- $PA 4$

(assuming O-strategy given by $[ ] f 3 4$).
Interpreting commands

The arena $\text{[ com ]}$

$\text{OQ} \quad \text{run}
\quad \vert
\text{PA} \quad \text{done}$

$\text{[ skip : com ]}$ is the P-strategy:

$\text{OQ} \quad \text{run}$

$\text{PA} \quad \text{done}$

Interpreting sequential composition: $\cdot$ com $\rightarrow$ com $\rightarrow$ com

com $\rightarrow$ com $\rightarrow$ com

$\text{OQ} \quad \text{run}$

$\text{PQ} \quad \text{run}$

$\text{OA} \quad \text{done}$

$\text{PQ} \quad \text{run}$

$\text{OA} \quad \text{done}$

$\text{PA} \quad \text{done}$
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\begin{align*}
OQ & \quad \text{run} \\
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Interpreting sequential composition: \(; : \text{com} \rightarrow \text{com} \rightarrow \text{com}\)

\[
\begin{align*}
\text{com} & \quad \rightarrow \quad \text{com} \quad \rightarrow \quad \text{com} \\
OQ & \quad \quad \quad \quad \text{run} \\
PQ & \quad \text{run} \\
OA & \quad \text{done} \\
PQ & \quad \text{run} \\
OA & \quad \text{done} \\
PA & \quad \quad \quad \quad \text{done}
\end{align*}
\]
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\mid \\
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**Interpreting sequential composition**: \(; : \text{com} \rightarrow \text{com} \rightarrow \text{com}\)

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\text{com} \rightarrow \text{com} \rightarrow \text{com} \\
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\]
Following Reynolds, we view a variable type as given (in an object-oriented style) by a product of its read method and its write method. Thus

$$\text{var} := \text{exp} \times \left( \prod_{i \in \omega} \text{com} \right)$$

- **read-part**: first component is the value held at that location
- **write-part**: second component contains countably many commands, namely, to write 0 (respectively 1, 2, etc.) to that location.

Thus arena $\llbracket \text{var} \rrbracket$ is the product arena $\llbracket \text{exp} \rrbracket \times \prod_{i \in \omega} \llbracket \text{com} \rrbracket$:

```
OQ  read  write(0)  write(1)  write(2)  \cdots
PA  0  1  2  \cdots  ok  ok  ok  \cdots
```
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\begin{array}{cccccc}
OQ & \text{read} & \text{write}(0) & \text{write}(1) & \text{write}(2) & \cdots \\
PA & 0 & 1 & 2 & \cdots & \text{ok} & \text{ok} & \text{ok} & \cdots
\end{array}
\]
Interpreting assignment: $X := M$

Write $X := M$ as $\text{assign } X M$. Thus

- $\text{assign} : \text{var} \rightarrow \text{exp} \rightarrow \text{com}$ is interpreted as a P-strategy
- context $[] X M$ determines an O-strategy

for playing in the arena $[\text{var} \rightarrow \text{exp} \rightarrow \text{com}]$.

\[
\begin{align*}
X & \rightarrow M \\
\text{var} & \rightarrow \text{exp} & \rightarrow \text{com} \\
OQ & \\
PQ & q \\
OA & 5 \\
PQ & \text{write}(5) \\
OA & \text{ok} \\
PA & \text{done}
\end{align*}
\]

(assuming O-strategy is given by context $[] X 5$)
Interpreting assignment: $X := M$

Write $X := M$ as **assign** $X M$. Thus

- **assign** : var → exp → com is interpreted as a P-strategy
- context $\mathbf{[]} X M$ determines an O-strategy for playing in the arena $\mathbf{[} \text{var} \rightarrow \text{exp} \rightarrow \text{com} \mathbf{]}$.

\[
\begin{array}{ccc}
X & \rightarrow & M \\
\text{var} & \rightarrow & \text{exp} & \rightarrow & \text{com} \\
OQ & \rightarrow & \text{run} \\
PQ & \rightarrow & q \\
OA & \rightarrow & 5 \\
PQ & \rightarrow & \text{write}(5) \\
OA & \rightarrow & \text{ok} \\
PA & \rightarrow & \text{done}
\end{array}
\]

(assuming O-strategy is given by context $\mathbf{[]} X 5$)
Interpreting block-allocated local variables: new

We decompose the formation

\[ \Gamma, x : \text{var} \vdash M : \text{com} \]

\[ \Gamma \vdash \text{new } x := n \text{ in } M : \text{com} \]

into two constructions:

1. **Currying**: \( \Gamma \vdash \lambda x : \text{var}. M : \text{var} \rightarrow \text{com} \)
2. **Application by a constant**: \( \text{new}_n : (\text{var} \rightarrow \text{com}) \rightarrow \text{com} \).

Thus we have

\[ \text{new } x := n \text{ in } M := \text{new}_n (\lambda x : \text{var}. M) \]

Accordingly \( [\text{new } x := n \text{ in } M] \) is the composite

\[ [\Gamma] \xrightarrow{[\Gamma \vdash \lambda x : \text{var}. M]} (\text{var} \rightarrow \text{com}) \xrightarrow{\text{new}_n} \text{com} \]

**Question.** What is the strategy \( \text{new}_n \)?
We decompose the formation

$$\Gamma, x : \text{var} \vdash M : \text{com}$$

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Thus we have

$$\text{new } x := n \text{ in } M := \text{new}_n (\lambda x : \text{var}. M)$$

Accordingly $\llbracket \text{new } x := n \text{ in } M \rrbracket$ is the composite

$$\llbracket \Gamma \rrbracket \xrightarrow{\llbracket \Gamma \vdash \lambda x : \text{var}. M \rrbracket} (\text{var} \rightarrow \text{com}) \xrightarrow{\text{new}_n} \text{com}$$

**Question.** What is the strategy $\text{new}_n$?
The strategy \( \text{new}_n : (\text{var} \rightarrow \text{com}) \rightarrow \text{com} \)

The plays in \( \text{new}_n \) should correspond to the behaviour of a \textit{prima facie} variable (initialized to \( n \)). Namely, they should satisfy:

**Good Variable Property**

Whenever the variable is read, it yields the value last written to it.

Thus, the (maximal) plays are defined to be words matching the regular expression:

\[
q \cdot q^{\langle 1 \rangle} \cdot (\text{read} \cdot n)^* \cdot \left( \sum_{i \geq 0} \text{write}(i) \cdot \text{ok} \cdot (\text{read} \cdot i)^* \right)^* \cdot \text{done}^{\langle 1 \rangle} \cdot \text{done}
\]

The (infinite) alphabet is the move-set of \( (\text{var} \rightarrow \text{com}^{\langle 1 \rangle}) \rightarrow \text{com} \) (subject to the labelling convention to distinguish copies of the same subarena).
Good Variable Behaviour: An Example Play in $\text{new}_n$

$$(\text{var} \rightarrow \text{com}^{(1)}) \rightarrow \text{com}$$

$OQ$ \hspace{2cm} \text{run}

$PQ$ \hspace{2cm} $\text{run}^{(1)}$

$OQ$ \hspace{2cm} \text{read}$

$PA$ \hspace{2cm} $n$

$OQ$ \hspace{2cm} \text{write}(5)$

$PA$ \hspace{2cm} \text{ok}$

$OQ$ \hspace{2cm} \text{read}$

$PA$ \hspace{2cm} $9$

$OA$ \hspace{2cm} $\text{done}^{(1)}$

$PA$ \hspace{2cm} $\text{done}$
1. Idealized Algol and Observational Equivalence

2. Game Semantics: An Impressionistic Introduction

3. Using Game Semantics to Decide Observational Equivalence

4. Homer: Higher-order Observational-equivalence Model checking
Recall: \[ \text{ord}(b) := 0 \quad \text{ord}(T_1 \rightarrow T_2) := \max(\text{ord}(T_1) + 1, \text{ord}(T_2)) \]

An \(\text{IA}_f\)-term \(x_1 : T_1, \ldots, x_n : T_n \vdash M : T\) is an \(i\)-th order term just if \(\text{ord}(T_j) < i\) and \(\text{ord}(T) \leq i\).

- \(\text{IA}_i\): collection of \(i\)-th order \(\text{IA}_f\)-terms.
- \(\text{IA}_i + \text{while}\) is \(\text{IA}_i\) augmented by while-loops.
- \(\text{IA}_i + Y_j\) (where \(j < i\)) is \(\text{IA}_i\) augmented by
  
  \[
  \Gamma, f : T \vdash M : T \\
  \Gamma \vdash \mu f^T.M : T
  \]

  where the premise is \(i\)-th order, and \(\text{ord}(T) \leq j\).

  I.e. \(\text{IA}_i + Y_j\) consists of \(\text{IA}_i\) and recursively-defined terms of order at most \(j\).
Theorem (Full Abstraction, Abramsky + McCusker 1997)

Observational equivalence of $\lambda\mu$ is characterized by complete plays (i.e. plays ending with a move that answers the opening question):

$$M \approx N \iff \text{cplays}(\llbracket \Gamma \vdash M : A \rrbracket) = \text{cplays}(\llbracket \Gamma \vdash N : A \rrbracket)$$

At low types, game semantics admits a concrete representation.

Theorem (Ghica + McCusker 2000)

In $\lambda\mu_{\ast}+$while:

1. $\text{cplays}(\llbracket \Gamma \vdash M : A \rrbracket)$ is regular. Further
2. $\text{cplays}(\llbracket \Gamma \vdash M : A \rrbracket)$, given as a DFA (or regular expression), can be constructed by recursion over syntax.

Hence $\approx$ in $\lambda\mu_{\ast}$ reduces to the problem of DFA-equivalence.
First steps in Algorithmic Game Semantics

Theorem (Full Abstraction, Abramsky + McCusker 1997)

Observational equivalence of IA is characterized by complete plays (i.e. plays ending with a move that answers the opening question):

\[ M \approx N \iff \text{cplays}(\llbracket \Gamma \vdash M : A \rrbracket) = \text{cplays}(\llbracket \Gamma \vdash N : A \rrbracket) \]

At low types, game semantics admits a concrete representation.

Theorem (Ghica + McCusker 2000)

In IA₂+\textbf{while}:

1. \text{cplays}(\llbracket \Gamma \vdash M : A \rrbracket) is regular. Further
2. \text{cplays}(\llbracket \Gamma \vdash M : A \rrbracket), given as a DFA (or regular expression), can be constructed by recursion over syntax.

Hence \approx in IA₂ reduces to the problem of DFA-equivalence.
**Observation (Obs Equiv)**: Given $\beta$-nfs $M$ and $N$ in sublanguage $L$ of IA, does $M \approx N$?

<table>
<thead>
<tr>
<th>$\text{IA}_i$</th>
<th>pure</th>
<th>$+\text{while}$</th>
<th>$+Y_0$</th>
<th>$+Y_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{IA}_0$</td>
<td>PTIME</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>$\text{IA}_1$</td>
<td>coNP</td>
<td>PSPACE</td>
<td>DPDA Equiv</td>
<td>undecidable</td>
</tr>
<tr>
<td>$\text{IA}_2$</td>
<td>PSPACE</td>
<td>PSPACE</td>
<td>DPDA Equiv</td>
<td>undecidable</td>
</tr>
<tr>
<td>$\text{IA}_3$</td>
<td>EXPTIME</td>
<td>EXPTIME</td>
<td>DPDA Equiv</td>
<td>undecidable</td>
</tr>
<tr>
<td>$\text{IA}_i$, $i \geq 4$</td>
<td>undecidable</td>
<td>undecidable</td>
<td>undecidable</td>
<td>undecidable</td>
</tr>
</tbody>
</table>

Undecidability results: O. LICS’02 and Murawski LICS’03.

**coNP + PSPACE** results: Murawski TCS 2005.

**EXPTIME** results: O. LICS’02; Murawski + Walukiewicz FOSSACS’05.

**DPDA Equiv** results: Murawski, Walukiewicz + O. ICALP’05.
Deciding $\approx$ for $\text{IA}_3 + \text{while}$ is EXPTIME-complete \textbf{ (M.+W.’05)}

**Visibly pushdown automata** (Alur+Madhusudan, STOC’04)

- Stack action is determined by input alphabet read:
  \[ \Sigma = \Sigma_{\text{push}} + \Sigma_{\text{pop}} + \Sigma_{\text{noop}} \]
- Excellent closure properties (almost as good as regular languages)

**VPA-languages**: Closed under complementation and intersection (cf. DPDA).

\[ L(A) \subseteq L(B) \iff L(A) \cap L(B) = \emptyset \]

is EXPTIME-complete, and in PTIME if $B$ deterministic.

**Theorem (Murawski+Walukiewicz 2005)**

\textit{The complete plays of ($\text{IA}_3 + \text{while}$)-terms are VPA-recognizable.}

**EXPTIME-hardness**: by reducing the EXPTIME-complete problem
\textbf{Finite Tree Automata Equivalence} (Seidl 1990) to it.
Deciding $\approx$ for $IA_i + Y_0$ (for $i = 1, 2, 3$) is equivalent to DPDA-Equiv

[Murawski, O. + Walukiewicz ICALP’05]

$IA_i + Y_0$: Only terms of base type can call themselves recursively. This includes all tail-recursive functions (i.e. iterations) and:

**Example.** Non tail-recursive ground recursion:

$$c : \text{com}, b : \text{bool} \vdash \mu p^{\text{com}}. \text{if } b \text{ then } (p ; c ; p) \text{ else skip} : \text{com}$$

**DPDA-Equivalenc hardness**

**Theorem**

There is a translation that maps a DPDA $A$ to a $(IA_1 + Y_0)$-term $x : \text{exp} \vdash M_A : \text{com}$ such that for any $A, B$, we have $L(A) = L(B)$ iff $M_A \approx M_B$. 

Luke Ong (Oxford)
Outline

1. Idealized Algol and Observational Equivalence
2. Game Semantics: An Impressionistic Introduction
3. Using Game Semantics to Decide Observational Equivalence
4. Homer: Higher-order Observational-equivalence Model checkER
HOMER: Higher-order Observational-equivalence Model check ER

[Hopkins + O. CAV 2009]

HOMER: a prototype tool implementing Murawski and Walukiewicz’s algorithm.

HOMER maps $\text{IA}_3 + \text{while}$ terms to VPA representing the complete plays in their game semantics, then check for the equivalence of the VPA. If the input term is at most 2nd-order (possibly with iteration), the VPA-compile is just a DFA.

Counterexample. If the terms are inequivalent, HOMER will produce both a game-semantic and an operational-semantic counterexample, in the form of a play and a separating context respectively.

Property checking. HOMER can also model check a term against a regular property or LTL formula.

HOMER is written in about 8 KLOC of F#, including about 600 LOC for the VPA toolkit. It is the first model checker for third-order programs.
HOMER: a prototype tool implementing Murawski and Walukiewicz’s algorithm.

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Example 1

\[
x : \text{exp} \vdash x
\]

\[
x : \text{exp} \vdash \text{new } X \text{ in } (X := x ; \text{if } !X = 0 \text{ then } !X \text{ else } !X)
\]
Example 1'

\[ x : \ exp \ |- \ x \]

\[ x : \ exp \ |- \ \text{if } x = 0 \ \text{then } x \ \text{else } x \]
Example 2: Sorting algorithms

Why sorting?
“... it seems impossible to use Model Checking to verify that a sorting algorithm is correct since sorting correctness is a data-oriented property involving several quantifications and data structures.” [Bandera user manual]

Example: Equivalence of (respective implementations of) bubble sort and insertion sort.

- Program parameterized over array size ($n$) and basic data type ($\mathbb{Z}$ MOD 3).
- The DFA model is fully abstract. Only the actions of the non-local array are observable, and hence, represented.
- An array of size 20 (over integers MOD 3) has circa $3^{20}$ states (about 3.5 billion). Our model is highly abstract (though still accurate): it has only about 5500 states!
Conclusions and Further Directions

- Game semantics has clear operational content, while admitting **compositional methods** in the style of denotational semantics.
- The game-semantic approach to observational equivalence checking is **fully automatic, sound and complete**, and **compositional**.
- The model (given by complete plays) extracted is highly accurate, yet “compact”.
- To extend to infinite data types, use abstraction refinement techniques (Bakewell + Ghica, TACAS08) or prove auxiliary data-independence results.

Further directions

1. **Performance**: Exploit abstraction refinement techniques (CEGAR), and acceleration technologies (SMT-solvers) to improve scalability.
2. **Challenge of verifying highly structured programs** (e.g. object orientation / functional features e.g. Javascript, Perl, etc).
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